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Mr. Gmerek
Calculus
Problem Set 3.1

Derivative of a Function

1. What (from PS 2.4) is the slope of a curve y = f(x) at the point where x = a?

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   a. When this limit exists, it is called the **derivative of f at a**.

   b. The **derivative** of the function f with respect to the variable x is the function f' (read *f prime*) whose value at x is

   $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ provided the derivative exists.

   c. The domain of f' is the set of points in the domain of f for which the limit exists. This may be smaller than the domain of f.

      i. If f'(x) exists, we say that f **has a derivative (is differentiable)** at x. A function that is differentiable at every point of its domain is a **differentiable function**.

2. Differentiate $f(x) = x^2$.

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} 2x + h \xrightarrow[h \to 0]{} \boxed{2x}$$

   a. What is the slope of x^2 at x = 4, -3, and 0?

   8 at x = 4   0 at x = 0
   -6 at x = -3

3. Since h approaches 0 in $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, we can create an alternate definition of derivative, this time as x approaches a.

   a. The **derivative of** the function **f at the point x = a** is the limit $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided the limit exists.

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i. Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

$$\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \lim_{x \to a} \frac{x - a}{(x-a)(\sqrt{x} + \sqrt{a})} \xrightarrow{x \to a} \frac{1}{\sqrt{a} + \sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$$

ii. Rewrite your answer using x instead of a (since x approaches a), and voilà, you have found the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

iii. What is the domain of f'(x)?

$x > 0$

4. Derivative notations:

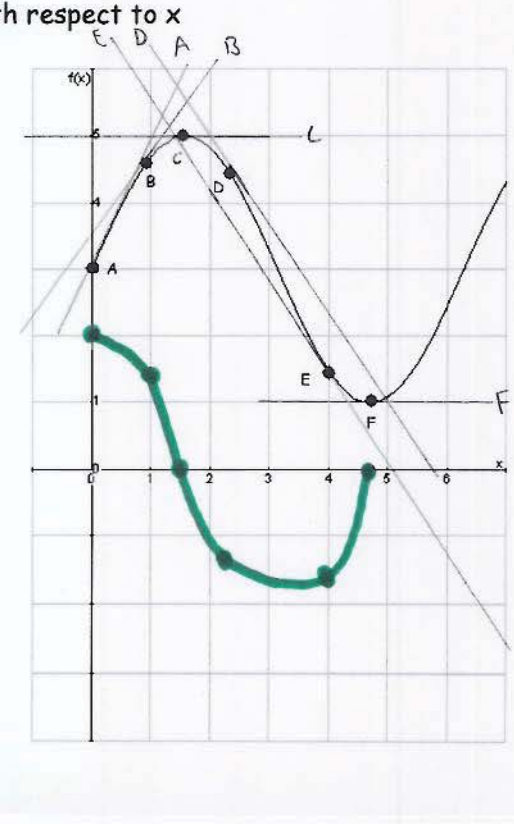
   a. y' – y prime

   b. $\dfrac{dy}{dx}$ - read "dy dx" – means the derivative of y with respect to x

   c. $\dfrac{df}{dx}$ - read "df dx" – means the derivative of f with respect to x

   d. $\dfrac{d}{dx}f(x)$ - means the derivative of f at x

5. Given the graph of f on the right, graph $f'(x)$ (Hint: How do we find the slope at a point on a graph?)



Draw tangent lines at each point and find the slope of the tangents.

$m_A = 2$ $\quad m_C = 0$ $\quad m_E = -1.5$

$m_B = 1.5$ $\quad m_D = -1.4$ $\quad m_F = 0$

   a. Discuss the behavior of f in terms of the signs and values of f'.

   * f is increasing when $f' > 0$
   * f is decreasing when $f' < 0$
   * When $f' = 0$, f has a horizontal tangent and changes from increasing to decreasing

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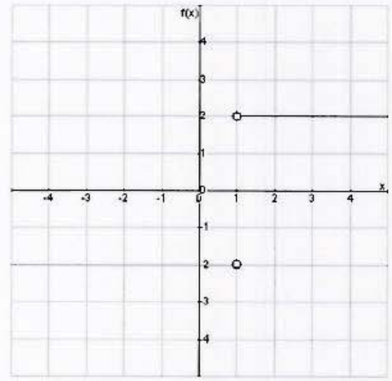
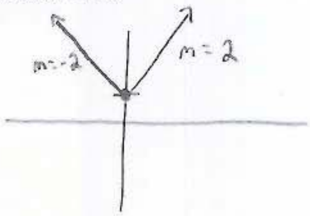
6. Sketch the graph of a function f that has the following properties.

   a. The graph of the derivative is as shown on the right.

   b. f(0) = 1

   c. f is continuous for all x





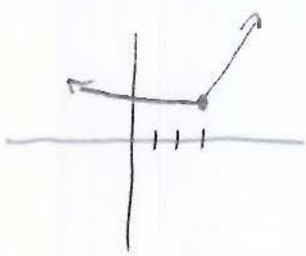
7. We now know that the derivative of a function is the same as ___Slope!___.

   a. How can we tell if a function is differentiable at a point?

If $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$

This means the slope is the same from both directions!

   b. Sketch a function that is not differentiable at x = 3 and explain why it is not differentiable.



This is not differentiable at $x=3$ because the slope as $x \to 3^-$ $\neq$ slope as $x \to 3^+$

   c. Is #6 differentiable at every x? Why or why not?

No, it is not differentiable at $x=0$ because

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

(once again, this means the slopes are not the same coming from both sides!)